Optimal Reconstruction Kernels in Medical Imaging

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Summary. In this paper we present techniques for deriving inversion algorithms in medical imaging. To this end we present a few imaging technologies and their mathematical models. They essentially consist of integral operators. The reconstruction is then recognized as the solution of an inverse problem. General strategies, the so-called approximate inverse, for deriving a solution are adapted. Results from real data are presented.

1 Introduction

The task in medical imaging is to provide in a non - invasive way information about the internal structure of the human body. The basic principle is that the patient is scanned by applying some sort of radiation and its interaction with the body is measured. This result is the data whose origin has to be identified. Hence we face an inverse problem.

There are several different imaging techniques and also different ways to characterize them. For the patient a very substantial difference is whether the source is inside or outside the body, whether we have *emission* or *transmission* tomography.

From the diagnostic point of view the resulting information is a way to distinguish the different techniques. Some methods provide information about the density of the tissue as x-ray computer tomography, ultrasound computer tomography, or diffuse tomography. A distinction between properties of the tissues is possible with magnetic resonance imaging and impedance computer tomography. Finally the localization of activities is possible with biomagnetism, (electrical activities), and emission computer tomography, (nuclear activities of injected pharmaceuticals).

From a physical point of view the applied wavelengths can serve as a classification. The penetration of electromagnetic waves into the body is sufficient only for wavelengths smaller than $10^{-11}m$ or larger than a few cm respectively. In the extremely short ranges are x rays, single particle emission tomography

and positron emission computer tomography. MRI uses wavelengths larger than $1m$; extremely long waves are used in biomagnetism. In the range of a few mm to a few cm are microwaves and ultrasound and light.

In this paper we present some principles in designing inversion algorithms in tomography. We concentrate on linear problems arising in connection with the Radon and the x-ray transform. In the original 2D x-ray CT problem the Radon transform served as mathematical model. Here one integrates over lines and the problem is to recover a function from its line integrals. The same holds in the 3D x-ray case, but in 3D the Radon transform integrates over planes, in general over $N-1$ - dimensional hyperplanes in $I\!R^N$. Hence here the so-called x-ray transform is the mathematical model. Further differences are in the parametrization of the lines. The 3D - Radon transform merely appears as tool to derive inversion formula. In the early days of MRI (magnetic resonance imaging), at those days called NMR, nuclear magnetic resonance, it served as a mathematical model, see for example Marr-Chen-Lauterbur [MCL81], but then, due to the limitations of computer power in those days one changed the measuring procedure and scanned the Fourier transform of the searched-for function in two dimensions. Nowadays the Radon transform reappeared, now in three and even four dimensions as mathematical model in EPRI (electron parametric resonance imaging) where spectral - spatial information is the goal, see e.g. Kuppusamy et al [KCSWZ95].

The paper is organized as following. We start with a general principle for reconstruction information from measured data, the so-called approximate inverse, see Louis [Lou96], Louis-Maass [LM90]. The well-known inversion of the Radon transform is considered a model case for inversion. Finally we consider a 3D x-ray problem and present reconstructions from real data.

2 Approximate Inverse as a Tool for Deriving Inversion Algorithms

The integral operators appearing in medical imaging are typically compact operators between suitable Hilbert spaces. The inverse operator of those compact operators with infinite dimensional range are not continuous, which means that the unavoidable data errors are amplified in the solution. Hence one has to be very careful in designing inversion algorithms. They have to balance the demand for highest possible accuracy and the necessary damping of the influence of the unavoidable data errors. From the theoretical point of view, exact inversion formulae are nice, but they do not take care of data errors. The way out of this dilemma is the use of approximate inversion formulas whose principles are explained in the following.

For approximating the solution of

$$Af = g$$

we apply the method of approximate inverse, see Louis [Lou96]. The basic idea works as follows: choose a so-called mollifier $e_\gamma(x,y)$ which, for a fixed reconstruction point x, is a function of the variable y and which approximates the delta distribution for the point x. The parameter γ acts as regularization parameter. Simply think in the case of one spatial variable x of

$$e_\gamma(x,y) = \frac{1}{2\gamma}\chi_{[x-\gamma,x+\gamma]}(y)$$

where χ_Ω denotes the characteristic function of Ω. Then the mollifier fulfills

$$\int e_\gamma(x,y)dy = 1 \tag{1}$$

for all x and the function

$$f_\gamma(x) = \int f(y)e_\gamma(x,y)dy$$

converges for $\gamma \to 0$ to f. The larger the parameter γ the larger is the interval where the averaging takes place and hence the stronger is the smoothing. Now solve for fixed reconstruction point x the auxiliary problem

$$A^*\psi_\gamma(x,\cdot) = e_\gamma(x,\cdot) \tag{2}$$

where $e_\gamma(x,\cdot)$ is the chosen approximation to the delta distribution for the point x, and put

$$\begin{aligned} f_\gamma(x) &= \langle f, e_\gamma(x,\cdot)\rangle \\ &= \langle f, A^*\psi_\gamma(x,\cdot)\rangle = \langle Af, \psi_\gamma(x,\cdot)\rangle = \langle g, \psi_\gamma(x,\cdot)\rangle \\ &=: S_\gamma g(x)\,. \end{aligned}$$

The operator S_γ is called the approximate inverse and ψ_γ is the reconstruction kernel. To be precise it is the approximate inverse for approximating the solution f of $Af = g$. If we choose instead of e_γ fulfilling (2.1) a wavelet, then f_γ can be interpreted as a wavelet transform of f. Wavelet transforms are known to approximate in a certain sense derivatives of the transformed function f, see [LMR97]. Hence this is a possibility to find jumps in f as used in contour reconstructions, see [LM93, Lou96].
The advantage of this method is that ψ_γ can be pre-computed independently of the data. Furthermore, invariances and symmetries of the operator A^* can directly be transformed into corresponding properties of S_γ as the following consideration shows, see Louis [Lou96]. Let T_1 and T_2 be two operators intertwining with A^*

$$A^*T_2 = T_1A^*\,.$$

If we choose a standard mollifier E and solve $A^*\Psi = E$ then the solution of Eq. (2) for the special mollifier $e_\gamma = T_1E$ is given as

$$\psi_\gamma = T_2 \Psi \,.$$

As an example we mention, that if A^* is translation invariant; i.e., $T_1 f(x) = T_2 f(x) = f(x-a)$, then also the reconstruction kernel is translation invariant.

Sometimes it is easier to cheque these conditions for A itself. Using $AT_1^* = T_2^* A$ we get the above relations by using the adjoint operators.

This method is presented in [Lou99] as general regularization scheme to solve inverse problems. Generalizations are also given. The application to vector fields is derived by Schuster [Sch00].

If the auxiliary problem is not solvable then its minimum norm solution leads to the minimum norm solution of the original problem.

3 Inversion of the Radon Transform

We apply the above approach to derive inversion algorithms for the Radon transform. This represents a typical behaviour for all linear imaging problems. The Radon transform in $I\!R^N$ is defined as

$$\mathbf{R} f(\theta, s) = \int\limits_{I\!R^N} f(x) \delta(s - x^\top \theta)\, dx$$

for unit vectors $\theta \in S^{N-1}$ and $s \in I\!R$. Its inverse is

$$\mathbf{R}^{-1} = c_N \mathbf{R}^* I^{1-N} \tag{3}$$

where $\mathbf{R}^*$ is the adjoint operator from L_2 to L_2, also called the backprojection, defined as

$$\mathbf{R}^* g(x) = \int_{S^{N-1}} g(\theta, x^\top \theta) d\theta \,,$$

I^α is the Riesz potential defined via the Fourier transform as

$$\widehat{(I^\alpha g)}(\xi) = |\xi|^{-\alpha} \widehat{g(\xi)},$$

acting on the second variable of $\mathbf{R}f$ and the constant

$$c_N = \frac{1}{2} (2\pi)^{1-N} \,.$$

see e.g. [Nat86]. We start with a mollifier $e_\gamma(x, \cdot)$ for the reconstruction point x and get

$$\begin{aligned}\mathbf{R}^*\psi_\gamma(x,\cdot) &= e_\gamma(x,\cdot)\\ &= c_N\mathbf{R}^*I^{1-N}\mathbf{R}e_\gamma(x,\cdot)\end{aligned}$$

leading to

$$\psi_\gamma(x;\theta,s) = c_N I^{1-N}\mathbf{R}e_\gamma(x;\theta,s)\,.$$

The Radon transform for fixed θ is translational invariant; i.e., if we denote by $\mathbf{R}_\theta f(s) = \mathbf{R}f(\theta,s)$, then

$$\mathbf{R}_\theta T_1^a f = T_2^{a^\top\theta}\mathbf{R}_\theta f$$

with the shift operators $T_1^a f(x) = f(x-a)$ and $T_2^t g(s) = g(s-t)$. If we chose a mollifier $\bar{e}_\gamma$ supported in the unit ball centred around 0 that is shifted to x as

$$e_\gamma(x,y) = 2^{-N}\bar{e}_\gamma(\frac{x-y}{2})$$

then also e_γ is supported in the unit ball and the reconstruction kernel fulfills

$$\psi_\gamma(x;\theta,s) = \frac{1}{2}\bar{\psi}_\gamma(\theta,\frac{s-x^\top\theta}{2})$$

as follows from the general theory in [Lou96] and as was used for the 2D case in [LS96].
Furthermore, the Radon transform is invariant under rotations; i.e.,

$$\mathbf{R}T_1^U = T_2^U\mathbf{R}$$

for the rotation $T_1^U f(x) = f(Ux)$ with unitary U and ${T_2}^U g(\theta,s) = g(U\theta,s)$. If the mollifier is invariant under rotation; i.e.,

$$\bar{e}_\gamma(x) = \bar{e}_\gamma(\|x\|)$$

then the reconstruction kernel is independent of θ leading to the following observation.

Theorem 1. *Let the mollifier $e_\gamma(x,y)$ be of the form*

$$e_\gamma(x,y) = 2^{-N}\bar{e}_\gamma(\|x-y\|/2)$$

then the reconstruction kernel is a function only of the variable s and the algorithm is of filtered backprojection type

$$f_\gamma(x) = \int_{S^{n-1}}\int_{I\!R}\psi_\gamma(x^\top\theta - s)\mathbf{R}f(\theta,s)dsd\theta\,. \tag{4}$$

First references to this technique can be found in the work of Grünbaum [DG81] and Solmon, [HS88].

Lemma 1. *The function f_γ from Theorem 3.1 can be represented as a smoothed inversion or as a reconstruction of smoothed data as*

$$f_\gamma = \mathbf{R}_\gamma^{-1} g = M_\gamma \mathbf{R}^{-1} g = \mathbf{R}^{-1} \tilde{M} g \tag{5}$$

where

$$M_\gamma f(x) = \langle f, e_\gamma(x, \cdot) \rangle$$

and

$$\tilde{M}_\gamma g(\theta, s) = \int_{I\!R} g(\theta, t) \tilde{e}_\gamma(s - t) dt$$

where

$$\tilde{e_\gamma}(s) = \mathbf{R} e_\gamma(s)$$

for functions e_γ fulfilling the conditions of Theorem 3.1.

4 Optimality Criteria

There are several criteria which have to be optimized. The speed of the reconstruction is an essential issue. The scanning time has to be short for the sake of the patients. In order to guarantee a sufficiently high patient throughput the time for the reconstruction cannot slow down the whole system, it has to be achieved in real - time. The above mentioned invariances adapted to the mathematical model give acceptable results. But the speed itself is not sufficient, the accuracy has to be best possible to ensure the medical diagnosis. This accuracy is determined by the number of data and the amount of unavoidable noise in the data.

To optimise with respect of accuracy and noise reduction we consider the problem in suitable Sobolev spaces $H^\alpha = H^\alpha(I\!R^N)$

$$H^\alpha = \{ f \in S' : \|f\|_{H^\alpha}^2 = \int_{I\!R^N} (1 + |\xi|^2)^\alpha |\hat{f}(\xi)|^2 d\xi < \infty \}$$

The corresponding norm on the cylinder $C^N = S^{N-1} \times I\!R$ is evaluated as

$$\|g\|_{H^\alpha(C^N)}^2 = \int_{S^{N-1}} \int_{I\!R} (1 + |\sigma|^2)^\alpha |\hat{g}(\theta, \sigma)|^2 d\sigma d\theta$$

where the Fourier transform is computed with respect to the second variable. We make the assumption that there is a number $\alpha > 0$ such that

$$c_1 \|f\|_{-\alpha} \leq \|Af\|_{L_2} \leq c_2 \|f\|_{-\alpha}$$

for all $f \in N(A)^\perp$. For the Radon transform in $I\!R^N$ this holds with $\alpha = (N-1)/2$, see e.g. [Nat86, Lou84].

We assume the data to be corrupted by noise; i.e.,

$$g^\varepsilon = \mathbf{R}f + n$$

where the true solution

$$f \in H^\beta$$

and the noise

$$n \in H^t$$

with $t \leq 0$. In the case of white noise, characterized by equal intensity at all frequencies, see e.g. [Kuo, Lou89], we hence have $|\hat{n}(\theta, \sigma)| = \text{const}$ and this leads to $n \in H^t$ with $t < -1/2$.

As mollifier we select a low-pass filter in the Fourier domain, resulting in two dimensions in the so-called RAM - LAK - filter. Its disadvantages are described in the next section. The theoretical advantage is that we get information about the frequencies in the solution and hence the achievable resolution.

This means, we select a cut-off $1/\gamma$ for γ sufficiently small and

$$\hat{\tilde{e}}_\gamma(\sigma) = (2\pi)^{-1/2}\chi_{[-1/\gamma,1/\gamma]}(\sigma)$$

where χ_A denotes the characteristic function of A.

Theorem 2. *Let the true solution be $f \in H^\beta$ with $\|f\|_\beta = \rho$ and the noise be $n \in H^t(C^N)$ with $\|n\|_t = \varepsilon$.*

Then the total error in the reconstruction is for $s < \beta$

$$\|\mathbf{R}_\gamma^{-1} g^\varepsilon - f\|_s \leq c\|n\|_t^{(\beta-s)/(\beta-t+(N-1)/2)} \|f\|_\beta^{(s-t+(N-1)/2)/(\beta-t+(N-1)/2)} \quad (6)$$

when the cut-off frequency is chosen as

$$\gamma = \eta \left(\frac{\|n\|_t}{\|f\|_\beta} \right)^{1/(\beta-t+(N-1)/2)} \quad (7)$$

Proof. We split the error in the data error and the approximation error as

$$\|\mathbf{R}_\gamma^{-1} g^\varepsilon - f\|_s \leq \|\mathbf{R}_\gamma^{-1} n\|_s + \|\mathbf{R}_\gamma^{-1}\mathbf{R}f - f\|_s.$$

In order to estimate the data error we introduce polar coordinates and apply the so-called projection theorem

$$\hat{f}(\sigma\theta) = (2\pi)^{(1-N)/2}\widehat{\mathbf{R}f}(\theta, \sigma) \quad (8)$$

relating Radon and Fourier transform. With $\widehat{\tilde{M}_\gamma g} = (2\pi)^{1/2}\hat{\tilde{e}}_\gamma \hat{g}$ we get

$$\begin{aligned}
\|\mathbf{R}_\gamma^{-1} n\|_s^2 &= (2\pi)^{1-N} \int_{S^{N-1}} \int_{I\!R} (1+|\sigma|^2)^s \sigma^{N-1} |\mathbf{R}\widehat{\mathbf{R}_\gamma^{-1}} n|^2 d\sigma d\theta \\
&= (2\pi)^{1-N} \int_{S^{N-1}} \int_{I\!R} (1+|\sigma|^2)^{s-t} \sigma^{N-1} (1+|\sigma|^2)^t |\widehat{\tilde{M}_\gamma n}|^2 d\sigma d\theta \\
&\leq (2\pi)^{1-N} \sup_{|\sigma|\leq 1/\gamma} ((1+|\sigma|^2)^{s-t} |\sigma|^{N-1}) \|n\|_t^2 \\
&= (2\pi)^{1-N} (1+\gamma^{-2})^{s-t} \gamma^{1-N} \|n\|_t^2 \\
&\leq (2\pi)^{1-N} 2^{s-t} \gamma^{2(t-s)+1-N} \|n\|_t^2
\end{aligned}$$

where we have used $\gamma \leq 1$. Starting from $\hat{\hat{e}}_\gamma = \mathbf{R}e_\gamma$ we compute the Fourier transform of e_γ via the projection theorem as $\hat{e}_\gamma(\xi) = (2\pi)^{-N}\chi_{[0,1/\gamma]}(|\xi|)$ and compute the approximation error as

$$\begin{aligned}
\|\mathbf{R}_\gamma^{-1}\mathbf{R}f - f\|_s &= \int_{I\!R^N} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \\
&= \int_{|\xi|\geq 1/\gamma} (1+|\xi|^2)^{(s-\beta)}(1+|\xi|^2)^\beta |\hat{f}(\xi)|^2 d\xi \\
&\leq \sup_{|\xi|\geq 1/\gamma} (1+|\xi|^2)^{(s-\beta)} \|f\|_\beta^2 \\
&\leq \gamma^{2(\beta-s)} \|f\|_\beta^2
\end{aligned}$$

The total error is hence estimated as

$$\|\mathbf{R}_\gamma^{-1} g^\varepsilon - f\|_s \leq (2\pi)^{(1-N)/2} 2^{(s-t)/2} \gamma^{(t-s)+(1-N)/2} \|n\|_t + \gamma^{(\beta-s)} \|f\|_\beta$$

Next we minimize this expression with respect to γ where we put with $a = s - t + (N-1)/2$ and

$$\varphi(\gamma) = c_1 \gamma^{-a} \varepsilon + \gamma^{\beta-s} \rho$$

Differentiation leads to the minimum at

$$\gamma = \left(\frac{c_1 a \varepsilon}{(\beta - s)\rho} \right)^{1/(\beta - s + a)}$$

Inserting in φ completes the proof.

This result shows, that if the data error goes to zero, the cut-off goes to infinity. It is related to the inverse of the signal-to-noise ratio.

5 The Filtered Backprojection for the Radon Transform in 2 and 3 Dimensions

In the following we describe the derivation of the filtered backprojection, see Theorem 3.1, for two and three dimensions. As seen in Formula (3.1) the inverse operator of the Radon transform in $I\!R^N$ has the representation

$$\mathbf{R}^{-1} = \mathbf{R}^* B$$

with

$$B = c_N I^{1-N}.$$

Hence using

$$e = \mathbf{R}^{-1}\mathbf{R}e = \mathbf{R}^* B \mathbf{R} e = \mathbf{R}^* \psi$$

this can easily be solved as

$$\psi_\gamma = c_N I^{1-N} \mathbf{R} e_\gamma. \tag{9}$$

As mollifier we choose a translational and rotational invariant function

$$\bar{e}_\gamma(x, y) = e_\gamma(\|x - y\|)$$

whose Radon transform then is a function of the variable s only. Taking the Fourier transform of Equation (4.1) we get

$$\begin{aligned}\hat{\psi}_\gamma(\sigma) &= c_N(I^{1-N}\widehat{(\mathbf{R}e_\gamma)})(\sigma) \\ &= \frac{1}{2}(2\pi)^{(1-N)/2}|\sigma|^{N-1}\hat{e}_\gamma(\sigma),\end{aligned}$$

where in the last step we have used again the projection theorem

$$\hat{f}(\sigma\theta) = (2\pi)^{(1-N)/2}\widehat{\mathbf{R}_\theta f}(\sigma).$$

So, we can proceed in the following two ways. Either we prescribe the mollifier e_γ, where the Fourier transform is then computed to

$$\hat{e}_\gamma(\sigma) = \sigma^{1-N/2}\int_0^\infty e_\gamma(s)s^{N/2}J_{N/2-1}(s\sigma)ds$$

where J_ν denotes the Bessel function of order ν. On the other hand we prescribe

$$\hat{e}_\gamma(\sigma) = (2\pi)^{-N/2}F_\gamma(\sigma)$$

with a suitably chosen filter F_γ leading to

$$\hat{\psi}_\gamma(\sigma) = \frac{1}{2}(2\pi)^{1/2-N}|\sigma|^{N-1}F_\gamma(\sigma).$$

If F_γ is the ideal low-pass; i.e., $F_\gamma(\sigma) = 1$ for $|\sigma| \le \gamma$ and 0 otherwise, then the mollifier is easily computed as

$$e_\gamma(x, y) = (2\pi)^{-N/2}\gamma^N \frac{J_{N/2}(\gamma\|x - y\|)}{(\gamma\|x - y\|)^{N/2}}.$$

In the two-dimensional case the calculation of ψ leads to the so called RAMLAK filter, which has the disadvantage to produce ringing artefacts due to the discontinuity in the Fourier domain.

More popular for 2D is the filter

$$F_\gamma(\sigma) = \begin{cases} \operatorname{sinc}\frac{\sigma\pi}{2\gamma}\ , & |\sigma| \le \gamma, \\ 0 \qquad\quad , & |\sigma| > \gamma \end{cases}$$

From this we compute the kernel ψ_γ by inverse Fourier transform to get for $\gamma = \pi/h$ where h is the stepsize on the detector; i.e., $h = 1/q$ if we use $2q + 1$ points on the interval $[-1, 1]$ and $s = s_\ell = \ell h$, $\ell = -q, \ldots, q$

$$\psi_\gamma(s_\ell) = \frac{\gamma^2}{\pi^4}\frac{1}{1-4\ell^2}\ ,$$

known as Shepp - Logan kernel.

The algorithm of filtered backprojection is a stable discretization of the above described method using the composite trapezoidal rule for computing the discrete convolution. Instead of calculating the convolution for all points $\theta^\top x$ the convolution is evaluated for equidistant points ℓh and then a linear interpolation is applied. Nearest neighbour interpolation is not sufficiently accurate, higher order interpolation is not bringing any improvement because the interpolated functions are not smooth enough. Then the composite trapezoidal rule is used for approximating the backprojection. Here one integrates a periodic function, hence, as shown with the Euler- Maclaurin summation formula, this formula is highly accurate. The filtered backprojection then consists of two steps. Let the data $\mathbf{R}f(\theta, s)$ be given for the directions $\theta_j = (\cos\varphi_j, \sin\varphi_j)$, $\varphi_j = \pi(j-1)/p$, $j = 1, ..., p$ and the values $s_k = kh$, $h = 1/q$ and $k = -q, ..., q$.

Step 1: For j=1,...,p, evaluate the discrete convolutions

$$v_{j,\ell} = h\sum_{k=-q}^{q}\psi_\gamma(s_\ell - s_k)\mathbf{R}f(\theta_j, s_k),\ \ell = -q, ..., q. \tag{10}$$

Step 2: For each reconstruction point x compute the discrete backprojection

$$\tilde{f}(x) = \frac{2\pi}{p}\sum_{j=1}^{p}\big((1-\eta)v_{j,\ell} + \eta v_{j,\ell+1} \tag{11}$$

where, for each x and j, ℓ and η are determined by

$$s = \theta_j^\top x, \ell \le s/h < \ell + 1, \eta = s/h - \ell$$

see e.g. [Nat86].

In the three - dimensional case we can use the fact, that the operator I^{-2} is local,

$$I^{-2}g(\theta, s) = \frac{\partial^2}{\partial s^2}g(\theta, s)$$

If we want to keep this local structure in the discretization we choose

$$F_\gamma(\sigma) = 2(1-\cos(h\sigma))/(h\sigma)^2$$

leading to

$$\psi_\gamma(s) = (\delta_\gamma - 2\delta_0 + \delta_{-\gamma})(s) \tag{12}$$

Hence, the application of this reconstruction kernel is nothing but the central difference quotient for approximating the second derivative. The corresponding mollifier then is

$$e_\gamma(y) = \begin{cases} (2\pi)^{-1}h^{-2}|y|^{-1} & \text{, for } |y| < h, \\ 0 & \text{, otherwise,} \end{cases}$$

see [Lou83]. The algorithm has the same structure as mentioned above for the 2D case.

In order to get reconstruction formulas for the fan beam geometry coordinate transforms can be used, the structure of the algorithms does not change.

6 Inversion Formula for the 3D Cone Beam Transform

In the following we consider the X-ray reconstruction problem in three dimensions when the data is measured by firing an X-ray tube emitting rays to a 2D detector. The movement of the combination source - detector determines the different scanning geometries. In many real - world applications the source is moved on a circle around the object. From a mathematical point of view this has the disadvantage that the data are incomplete, the condition of Tuy-Kirillov is not fulfilled. This condition says, that essentially the data are complete for the three - dimensional Radon transform. More precisely, all planes through a reconstruction point x have to cut the scanning curve Γ. We base our considerations on the assumptions that this condition is fulfilled, the reconstruction from real data nevertheless is then from the above described circular scanning geometry, because other data is not available to us so far.

A first theoretical presentation of the reconstruction kernel was given by Finch [Fin87], invariances were then used in the group of the author to speed-up the computation time considerably, so that real data could be handled, see [Lou03]. See also the often used algorithm from Feldkamp et al. [FDK84] and the contribution of Defrise and Clack [DC94]. The approach of Katsevich [Kat02] differs from our approach that he avoids the Crofton symbol by restricting the backprojection to a range dependent on the reconstruction point x. An overview of the so far existing reconstruction algorithms is given by [ZYW], it is based on a relation between the Fourier transform and the cone beam transform, derived by Tuy, [Tuy], generalizing the so-called projection theorem for the Radon transform, see Formula (4.3)

The presentation follows Louis [Lou06].

The mathematical model here is the so-called X-ray transform, where we denote with $a \in \Gamma$ the source position, where $\Gamma \subset \mathbb{R}^3$ is a curve, $\theta \in S^2$ is the direction of the ray:

$$\mathbf{D}f(a,\theta) = \int_0^\infty f(a+t\theta)dt$$

The adjoint operator of D as mapping from $L_2(\mathbb{R}^3) \longrightarrow L_2(\Gamma \times S^2)$ is given as

$$\mathbf{D}^* g(x) = \int_\Gamma |x-a|^{-2} g\left(a, \frac{x-a}{|x-a|}\right) da$$

Most attempts to find inversion formulae are based on a relation between X-ray transform and the 3D Radon transform, the so-called *Formula of Grangeat*, first published in Grangeat's PhD thesis [Gr87], see also [Gr91] :

$$\frac{\partial}{\partial s}\mathbf{R}f(\omega, a^\top\omega) = -\int_{S^2} \mathbf{D}f(a,\theta)\delta'(\theta^\top\omega)d\theta.$$

Proof. We copy the proof from [NW01]. It consists of the following two steps.
i) We apply the adjoint operator of $\mathbf{R}_\theta$

$$\int_{I\!R} \mathbf{R}f(\theta,s)\psi(s)ds = \int_{I\!R^3} f(x)\psi(x^\top\theta)dx$$

ii) Now we apply the adjoint operator $\mathbf{D}$ for fixed source position a

$$\int_{S^2} \mathbf{D}f(a,\theta)h(\theta)d\theta = \int_{I\!R^3} f(x)h\big(\frac{x-a}{|x-a|}\big)|x-a|^{-2}dx$$

Putting in the first formula $\psi(s) = \delta'(s - a^\top\omega)$ and use in the second $h(\theta) = \delta'(\theta^\top\omega)$ and the fact that δ' is homogeneous of degree -2 in $I\!R^3$ then this completes the proof.

We note the following rules for δ':
i)

$$\int_{S^2} \psi(a^\top\omega)\delta'(\theta^\top\omega)d\omega = -a^\top\theta \int_{S^2\cap\theta^\perp} \psi'(a^\top\omega)d\omega$$

ii)

$$\int_{S^2} \psi(\omega)\delta'(\theta^\top\omega)d\omega = -\int_{S^2\cap\theta^\perp} \frac{\partial}{\partial\theta}\psi(\omega)d\omega$$

Starting point is now the inversion formula for the 3D Radon transform

$$f(x) = -\frac{1}{8\pi^2}\int_{S^2} \frac{\partial^2}{\partial s^2}\mathbf{R}f(\omega, x^\top\omega)d\omega \tag{13}$$

rewritten as

$$f(x) = \frac{1}{8\pi^2}\int_{S^2}\int_{I\!R} \frac{\partial}{\partial s}\mathbf{R}f(\omega,s)\delta'(s - x^\top\omega)dsd\omega$$

We assume in the following that the Tuy-Kirillov condition is fulfilled. Then we can change the variables as: $s = a^\top\omega$, n is the Crofton symbol; i.e., the number of source points $a \in \Gamma$ such that $a^\top\omega = x^\top\omega$, $m = 1/n$ and get

$$
\begin{aligned}
f(x) &= \frac{1}{8\pi^2}\int_{S^2}\int_{\Gamma}(\mathbf{R}f)'(\omega, a^\top\omega)\delta'((a-x)^\top\omega)|a'^\top\omega|m(\omega,a^\top\omega)dad\omega \\
&= -\frac{1}{8\pi^2}\int_{S^2}\int_{\Gamma}\int_{S^2}\mathbf{D}f(a,\theta)\delta'(\theta^\top\omega)d\theta\delta'((a-x)^\top\omega)|a'^\top\omega|m(\omega,a^\top\omega)dad\omega \\
&= -\frac{1}{8\pi^2}\int_{\Gamma}|x-a|^{-2}\int_{S^2}\int_{S^2}\mathbf{D}f(a,\theta)\delta'(\theta^\top\omega)d\theta\delta'(\frac{(x-a)}{|x-a|}^\top\omega) \\
&\qquad \times|a'^\top\omega|m(\omega,a^\top\omega)dad\omega
\end{aligned}
$$

where we again used that δ' is homogeneous of degree -2. We now introduce the following operators

$$T_1 g(\omega) = \int_{S^2} g(\theta)\delta'(\theta^\top\omega)d\theta \tag{14}$$

and we use T_1 acting on the second variable as

$$T_{1,a}g(\omega) = T_1 g(a,\omega)\,.$$

We also use the multiplication operator

$$M_{\Gamma,a}h(\omega) = |a'^\top\omega|m(\omega,a^\top\omega)h(\omega)\,. \tag{15}$$

and state the following result.

Theorem 3. *Let the condition of Tuy-Kirillov be fulfilled. Then the inversion formula for the cone beam transform is given as*

$$f = -\frac{1}{8\pi^2}\mathbf{D}^* T_1 M_{\Gamma,a} T_1 \mathbf{D}f \tag{16}$$

with the adjoint operator $\mathbf{D}^*$ of the cone beam transform and T_1 and $M_{\Gamma,a}$ as defined above.

Note that the operators $\mathbf{D}^*$ and M depend on the scanning curve Γ.

This form allows for computing reconstruction kernels. To this end we have to solve the equation

$$D^*\psi_\gamma = e_\gamma$$

in order to write the solution of $\mathbf{D}f = g$ as

$$f(x) =< g, \psi_\gamma(x,\cdot) > \quad .$$

In the case of exact inversion formula e_γ is the delta distribution, in the case of the approximate inversion formula it is an approximation of this distribution, see the method of approximate inverse. Using that $\mathbf{D}^{-1} = -\frac{1}{8\pi^2}\mathbf{D}^* T_1 M_{\Gamma,a} T_1$ we get

$$\mathbf{D}^*\psi = \delta = -\frac{1}{8\pi^2}\mathbf{D}^* T_1 M_{\Gamma,a} T_1 \mathbf{D}\delta$$

and hence

$$\psi = -\frac{1}{8\pi^2} T_1 M_{\Gamma,a} T_1 \mathbf{D}\delta \tag{17}$$

We can explicitly give the form of the operators T_1 and $T_2 = MT_1$. The index at ∇ indicates the variable with respect to which the differentiation is performed.

$$\begin{aligned} T_1 g(a,\omega) &= \int_{S^2} g(a,\theta)\delta'(\theta^\top\omega)d\theta \\ &= -\omega^\top \int_{S^2\cap\omega^\perp} \nabla_2 g(a,\theta)d\theta \end{aligned}$$

and

$$\begin{aligned} T_1 M_{\Gamma,a} h(a,\alpha) &= \int_{S^2} \delta'(\omega^\top\alpha)|a'^\top\omega| m(\omega, a^\top\omega)h(a,\omega)d\omega \\ &= -a'^\top\alpha \int_{S^2\cap\alpha^\perp} \operatorname{sign}(a'^\top\omega) m(\omega,a^\top\omega)h(a,\omega)d\omega \\ &\quad -\alpha^\top \int_{S^2\cap\alpha^\perp} |a'^\top\alpha| \nabla_1 m(\omega,a^\top\omega)h(a,\omega)d\omega \\ &\quad -a^\top\alpha \int_{S^2\cap\alpha^\perp} |a'^\top\omega| \nabla_2 m(a,a^\top\omega)h(a,\omega)d\omega \\ &\quad -\int_{S^2\cap\alpha^\perp} |a'^\top\omega| m(\omega,a^\top\omega)\frac{\partial}{\partial\alpha}h(a,\omega)d\omega \end{aligned}$$

Note that the function m is piecewise constant, the derivatives are then Delta - distributions at the discontinuities with factor equal to the height of the jump; i.e., $1/2$.
Depending on the scanning curve Γ invariances have to be used. For the circular scanning geometry this leads to similar results as mentioned in [Lou03]. In the following we present a reconstruction from data provided by the Fraunhofer Institut for Nondestructive Testing (IzfP) in Saarbrücken. The detector size was $(204.8mm)^2$ with 512^2 pixels and 400 source positions on a circle around the object. The number of data is 10.4 million. The mollifier used is

$$e_\gamma(y) = (2\pi)^{-3/2}\gamma^{-3}\exp\Big(-\frac{1}{2}\Big|\frac{y}{\gamma}\Big|^2\Big).$$

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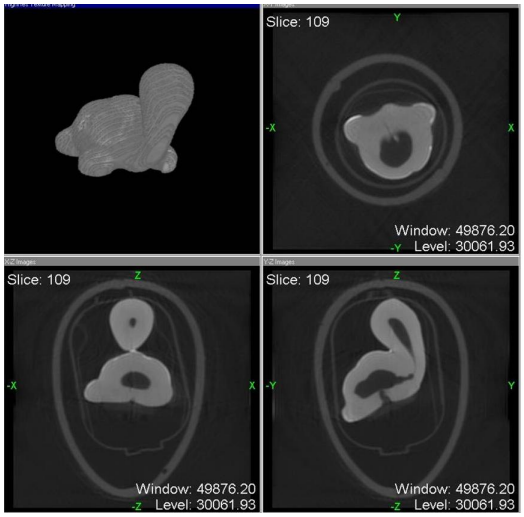

Fig. 1. Reconstruction of a surprise egg with a turtle inside.

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